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A note on D_p spaces

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In a recent paper [3] the author has introduced a new class of topological spaces, called D_p spaces. The purpose of this paper is to obtain some new characterizations of D_p spaces.

1. Preliminaries. Throughout the present paper, spaces will always mean topological spaces on which no separation axioms are assumed unless explicitely stated.

Definition 1.1. A space X is *paracompact* iff every open covering of X has an open locally finite refinement, [1].

Definition 1.2. Let X be a space and A a subset of X. The set A is *α-paracompact* iff every X-open cover of A has an X-open X-locally finite refinement which covers A, [8].

Definition 1.3. A subset A of a space X is *α-regular* iff for any point $a\in A$ and any X-open set containing a there exists an X-open set V such that $a\in V\subset \bar{V}\subset U$, [4].

Definition 1.4. A space X is D_p iff there exists an α-paracompact subset A such that $\bar{A}=X$, [3].

Theorem 1.1. ([3]) *Let X be a D_p space such that there exists a dense α-regular α-paracompact subset A. Then, every open covering of the set A has a closed locally finite refinement, hence every open covering of X has a locally finite closed refinement.*

Theorem 1.2. ([3]) *Let X be a space such that there is a dense α-regular subset D. If every X-open covering of D has an X-locally finite refinement which covers D, then every X-open covering of D has a closed (in X) X-locally finite refinement.*

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Theorem 1.3. ([3]) *Let X be a space such that there exists a dense α-regular subset D. Then if every X-open covering of D has an X-locally finite refinement which covers D, then D is α-paracompact, i.e. X is paracompact.*

Definition 1.5. An open cover $\mathscr{U}$ is *even* iff there exists a neighbourhood V of diagonal in $X \times X$ such that for each $x \in X$, $V(x) \subset U$ $(V(x) = \{y\colon (x, y) \in V\})$ for some $U \in \mathscr{U}$, [2].

Theorem 1.4. ([2]) *If the open covering $\mathscr{U}$ has a closed locally finite refinement, then $\mathscr{U}$ is even.*

Theorem 1.5. ([2]) *Let X be a space such that each open cover is even and let $\mathscr{A}$ be a locally finite (or a discrete) family of subsets of X. Then, there is an open neighbourhood V of the diagonal in $X \times X$ such that the family of all sets $V(A)$ $(V(A) = \cup \{V(x)\colon x \in A\})$ for A in $\mathscr{A}$ is locally finite (respectively discrete).*

Theorem 1.6. ([2]) *If every open covering of a space X is even, then any open cover of X has an open σ-discrete refinement.*

Definition 1.6. Let $\mathscr{A}$ be a family of subsets of a space X. The *star* of a point $x \in X$ in $\mathscr{A}$ is defined to be the union of all members of $\mathscr{A}$ which contain x. A family $\mathscr{A}$ of subsets of a space X is said to be *star refinement* of another family $\mathscr{B}$ of subsets of X iff the family of all stars of points of X in $\mathscr{A}$ forms a covering of X which refines $\mathscr{B}$.

Theorem 1.7. ([2]) *Every open covering of a space X is even iff every open covering has an open star refinement.*

Definition 1.7. A family $\mathscr{A}$ of subsets of a space X is called *closure preserving* iff for every subfamily $\mathscr{A}'$ of $\mathscr{A}$ we have $\cup \{\bar{A}\colon A \in \mathscr{A}'\} = \overline{\cup \{A\colon A \in \mathscr{A}'\}}$, [5].

Theorem 1.8. ([6]) *Let X be a space such that every open covering of X has a closure preserving closed refinement. Then:*

a) *X is normal;*

b) *Every open covering of X has a σ-discrete open refinement.*

Theorem 1.9. ([2]) *If every open covering of a space X has a σ-locally finite open refinement then, every open covering of X has a locally finite refinement.*

2. Main results.

Lemma 2.1. *Let D be any dense α-regular subset of a space X. If every X-open covering of D has an X-locally finite refinement which covers D, then every open covering of X has an open σ-discrete refinement.*

Proof. By assumption it follows that every open covering of D is open covering of X, hence by Theorem 1.4 it follows that every open covering of X is even. The result follows from Theorem 1.6.

Lemma 2.2. *Let D be any dense α-regular subset of a space X such that every open covering of D is open covering of X. Then, if every open covering of X has a σ-locally finite open refinement, then every open covering of X has a locally finite refinement, hence D is α-paracompact and X is paracompact.*

Proof. The result follows from Theorems 1.9 and 1.3.

Theorem 2.1. *Let D be any dense α-regular subset of a space X such that every open covering of D is open covering of X. Then, the following are equivalent:*

a) *X is paracompact;*
b) *D is α-paracompact;*
c) *every open covering of X has a locally finite closed refinement;*
d) *every open covering of X has a locally finite refinement;*
e) *every open covering of X is even;*
f) *every open covering of X has an open star refinement;*
g) *every open covering of X has a σ-discrete open refinement;*
h) *every open covering of X has a σ-locally finite open refinement.*

Proof. a)$\Leftrightarrow$b): Obvious.
b)$\Rightarrow$c): It follows from Theorem 1.1.
c)$\Rightarrow$d): Obvious.
d)$\Rightarrow$c): It follows from Theorem 1.2.
d)$\Rightarrow$a): It follows from Theorem 1.3.
c)$\Rightarrow$e): It follows from Theorem 1.4.
e)$\Leftrightarrow$f): It follows from Theorem 1.7.
e)$\Rightarrow$g): It follows from Lemma 2.1.
g)$\Rightarrow$h): Obvious.
h)$\Rightarrow$a): It follows from Lemma 2.2.

Corollary 2.1. *For a regular space, the following are equivalent:*

a) *X is paracompact;*
b) *every open covering of X has a locally finite closed refinement;*
c) *every open covering of X has locally finite refinement;*
d) *every open covering of X is even;*
e) *every open covering of X has an open star refinement;*
f) *every open covering of X has a σ-discrete open refinement;*
g) *every open covering of X has a σ-locally finite open refinement.*

The assumption "Every open covering of D is open covering of X" in Theorem 2.1 can not be dropped as can be seen from the following example.

Example 2.1. Let $X=\{a, b, a_i, b_i\colon i=1, 2, \dots\}$. Let each point a_i be isolated. Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a)\colon n=1, 2, \dots\} \quad \text{where} \quad V^n(a)=\{a, a_i\colon i \geqq n\}.$$

Let the fundamental system of neighbourhoods of b be the set

$$\{\{b\}\cup V^n(a)\colon n=1, 2, \dots\}.$$

Let the fundamental system of neighbourhoods of b_i be the set

$$\{U^n(b_i)\colon n=1, 2, \dots\} \quad \text{where} \quad U^n(b_i)=\{b_i, a_j\colon j \geqq n\}.$$

Let $D=\{a_i\colon i=1, 2, \dots\}$; D is α-regular. X is not regular at a, hence X is not regular. The subset D is not α-paracompact. X is not paracompact, since the family consisting of the sets

$$V^n(a), \ \{b\}\cup V^n(a), \ U^i(b_i) \quad \text{for all } i \text{ and all } \{a_i\}$$

is open covering of X which admits of no locally finite open refinement. The family consisting of the sets $\{a_i\}$ for all i is an X-open covering of D, but it is not open covering of X. Let

$$\mathcal{U}=\{U_i\colon i\in I\}$$

be any open covering of X. There exists n such that

$$\{b\}\cup V^n(a)\subset U_i$$

for some $U_i\in\mathcal{U}$. Let $\mathcal{V}_1$ be the family consisting of the sets

$$\{b\}\cup V^n(a), \ \{a_1\}, \{a_2\}, \dots, \{a_{n-1}\}.$$

For any b_i, there exists $n(b_i)$ such that $U^{n(b_i)}\subset U_{i(b_i)}$ for some $i(b_i)\in I$.
Let

$$\mathcal{V}=\{\mathcal{V}_1, U^{n(b_i)}\colon i=1, 2, \dots\}.$$

$\mathcal{V}$ is σ-locally finite open refinement of $\mathcal{U}$, but X is not paracompact i.e. D is not α-paracompact.

Lemma 2.3. *Let X be a space and D be a dense α-regular subset of X. If for every X-open covering $\mathcal{U}$ of D there exists a closure preserving family $\mathcal{V}$ which refines $\mathcal{U}$ and covers D, then for every X-open covering $\mathcal{A}$ of D there exists a closed closure preserving family $\mathcal{B}$ which refines $\mathcal{A}$ and covers D.*

Proof. Let $\mathcal{U}=\{U_i\colon i\in I\}$ be any X-open covering of D. Since D is α-regular, for each point $x\in D$, there exists an open set V_x such that $x\in V_x\subset\bar{V}_x\subset U_{i(x)}$ for some $i(x)\in I$. Let $\mathscr{V}=\{V_x\colon x\in D\}$. By assumption, there exists a closure preserving family

$$\mathscr{H}=\{H_j\colon j\in J\},$$

which refines $\mathscr{V}$ and covers D. Then $\{\overline{H}_j\colon j\in I\}$ is a closure preserving closed family which refines $\mathscr{U}$ and covers D.

From this lemma it follows that every open covering of D is an open covering of the space X.

Definition 2.1. A subset A of a space X is T_1 iff every point of A is closed in X.

Lemma 2.4. *Let D be a dense T_1 subset of a space X such that every X-open covering of D has a closed closure preserving refinement. Then, D is α-paracompact i.e. X is paracompact.*

Proof. From Theorem 1.8. it follows that X is normal i.e. D is α-regular. From Theorem 1.8 it follows that every open covering of X (by assumption it follows that every open covering of D is open covering of X) has a σ-discrete open refinement. Now, the result follows from Theorem 2.1.

Theorem 2.2. *Let D be a dense α-regular subset of a space X. Then, the following are equivalent:*

a) *D is α-paracompact;*

b) *every open covering of D has a closure preserving open refinement;*

c) *every open covering of D has a closure preserving refinement;*

d) *every open covering of D has a closure preserving closed refinement.*

Proof. a)$\Rightarrow$b): Every locally finite family is closure preserving.

b)$\Rightarrow$c): Obvious.

c)$\Rightarrow$d): It follows from Lemma 2.3.

d)$\Rightarrow$a): It follows from Lemma 2.4.

Corollary 2.2. ([6]) *For a regular space X, the following are equivalent:*

a) *X is paracompact;*

b) *every open covering of X has a closure preserving open refinement;*

c) *every open covering of X has a closure preserving refinement;*

d) *every open covering of X has a closure preserving closed refinement.*

Corollary 2.3. *Let D be a dense α-regular α-paracompact subset of X. Then, X is normal.*

Proof. From Theorem 2.2 it follows that every open covering of D (hence of X) has a closure preserving closed refinement, hence by Theorem 1.8 it follows that X is normal.

There exists a space with the properties as in Theorem 2.2 which is not regular. The following example will serve the purpose.

Example 2.2. Let $X=\{a, b, a_i: i=1, 2, \ldots\}$. Let each point a_i be isolated. Let $\{V^n(a): n=1, 2, \ldots\}$ be the fundamental system of neighbourhoods of a where $V^n(a)=\{a, a_i: i\geqq n\}$.

Let $\{U^n(b): n=1, 2, \ldots\}$ be the fundamental system of neighbourhoods of b where

$$U^n(b) = \{b, a, a_i: i \geqq n\}.$$

Let $D=\{b, a_i: i=1, 2, \ldots\}$; D is a dense T_1 (α-regular) α-paracompact subset of X. X is normal, X is not T_1. X is not regular at a, hence X is not regular.

Theorem 2.3. *Let D be a dense T_1 α-paracompact subset of a normal space X. If f is a closed and continuous mapping of the space X onto a space Y, then Y is paracompact.*

Proof. Let D be a dense T_1 α-paracompact subset of a normal space X. Y is normal. Since $f(\bar{D})=\overline{f(D)}=Y$, it follows that $f(D)$ is the dense T_1 (hence α-regular) subset of the normal space Y. Let $\mathcal{U}=\{U_i: i\in I\}$ be any open covering of $f(D)$. Let $\mathcal{W}=\{f^{-1}(U_i): U_i\in\mathcal{U}\}$, it is the open covering of D (hence it is open covering of X). It follows that every open covering of $f(D)$ is an open covering of Y. $\mathcal{W}$ has a closure preserving closed refinement $\mathcal{A}=\{A_j: j\in J\}$.

Then, $f(\mathcal{A})=\{f(A_j): j\in J\}$ is the closure preserving closed refinement of $\mathcal{U}$, hence Y is paracompact.

Corollary 2.4. ([6]) *The image of a Hausdorff paracompact space, under a continuous closed mapping, must be paracompact.*

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